Propagation of light in anisotropic media

As has been shown previously, when propagating in isotropic media, light simply follow refraction laws. When crossing the border with anisotropic media, penetration of light differs, usually causing the wave to split into different polarizations.

Demonstration

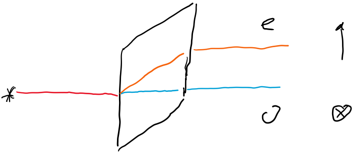
Icelandic spar

Vikings

Let's direct light onto the icelandic spar crystal.

Bartolinus Erasmus

(1670)



Ordinary wave follow Snell's law. Meanwhile, extraordinary changes direction even at normal incidence. This beams will be polarized in perpendicular directions.

Demonstration

We observed birefringence in natural and manufactured media.

Our goal is to understand the nature of birefringence. We also need to derive math that will allow us to determine direction of propagation in anisotropic media.

What is optical anisotropy?

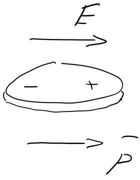
It is effect when optical properties depend on direction of light propagation.

Not all anisotropic media are optically anisotropic. NaCl is mechanically anisotropic, but optically isotropic.

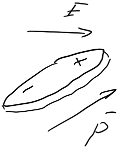
Reasons for optical anisotropy:

① Anisotropy of local oscillators

$\bar{p} \parallel \bar{E}$



$\bar{p} \nparallel \bar{E}$



That means electrons will oscillate differently in different directions along and perpendicular to the bond.

② Anisotropy of local oscillator positions.

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Most important, dipole moment of oscillator is not parallel to the field. $\bar{P} \nparallel \bar{E}$

$$\bar{p} = \varepsilon_0 \hat{a} \bar{E} \qquad \begin{cases} p_x = \varepsilon_0 (a_{xx} E_x + a_{xy} E_y + a_{xz} E_z) \\ p_y = \varepsilon_0 (a_{yx} E_x + a_{yy} E_y + a_{yz} E_z) \\ p_z = \varepsilon_0 (a_{zx} E_x + a_{zy} E_y + a_{zz} E_z) \end{cases}$$

a single molecule

$$p_i = \varepsilon_0 \alpha_{ij} E_j \qquad i = x, y, z$$

α_{ij} – polarizability tensor

Polarizability is a measure how simple we can move electron under the external field.

If we have N particles:

$$P_i = \varepsilon_0 N \alpha_{ij} E_j = \varepsilon_0 \chi_{ij} E_j$$

χ_{ij} – susceptibility tensor

Then we will employ connection between $\bar{D}$, $\bar{E}$ and $\bar{P}$.

$\bar{D} = \varepsilon_0 \bar{E} + \bar{P}$ $\quad D_i = \varepsilon_0 (\delta_{ij} + \chi_{ij}) E_j = \varepsilon_0 \varepsilon_{ij} E_j$

$\varepsilon_{ij} = \delta_{ij} + \chi_{ij}$ $\quad \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$ Kronecker delta

This tensor is

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{pmatrix}$$

This tensor is symmetrical

$\varepsilon_{ij} = \varepsilon_{ji}$

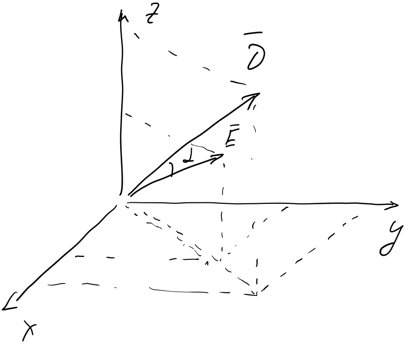
Since this tensor is symmetric, then we can diagonalize it:

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{pmatrix}$$

In anisotropic media all ε_x, ε_y and ε_z are different

$$\begin{cases} D_x = \varepsilon_0 \varepsilon_x E_x \\ D_y = \varepsilon_0 \varepsilon_y E_y \\ D_z = \varepsilon_0 \varepsilon_z E_z \end{cases}$$

That means vectors $\bar{E}$ and $\bar{D}$ are not parallel in such media.



Let's talk about propagation

Let's remind ourselfs with Maxwell equations.

$$[\bar\nabla \bar E] = -\frac{\partial \bar B}{\partial t} \qquad [\bar\nabla \bar H] = \frac{\partial \bar D}{\partial t}$$

$$\bar H \to \bar B$$

$$\bar H = \frac{\bar B}{\mu_0} = \frac{\bar B\, \varepsilon_0}{\mu_0 \varepsilon_0} = \varepsilon_0 c^2 \bar B$$

Please, remember, we consider non-magnetic media, hence $\mu = 1$.

Thus,

$$\begin{cases} [\bar\nabla \bar E] = -\dfrac{\partial \bar B}{\partial t} \\ \varepsilon_0 c^2 [\bar\nabla \bar B] = \dfrac{\partial \bar D}{\partial t} \end{cases}$$

Our main equations to work with

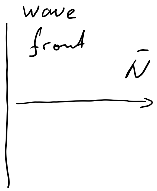
Let's now consider propagation of plane waves:

$$\begin{cases} \bar{E} = \bar{E}_0 e^{i(\omega t - \bar{k}\bar{r})} \\ \bar{B} = \bar{B}_0 e^{i(\omega t - \bar{k}\bar{r})} \\ D = D_0 e^{i(\omega t - \bar{k}\bar{r})} \end{cases}$$

Let's introduce elementary vector $\bar{N} = \frac{\bar{k}}{k} = \frac{v}{\omega}\bar{k}$

v - phase velocity

$\bar{N}$ is parallel to $\bar{k}$, hence perpendicular to wave front.



As we already know

$$\begin{cases} \bar{\nabla} \to i\bar{k} \\ \frac{\partial}{\partial t} \to i\omega \end{cases}$$

We can simplify our main equations

1) $\varepsilon_0 c^2 [\bar\nabla \bar B] = \frac{\partial \bar D}{\partial t}$

$$\varepsilon_0 c^2 i [\bar k \bar B] = i\omega \bar D \qquad \boxed{\varepsilon_0 c^2 [\bar N \bar B] = -v \bar D}$$

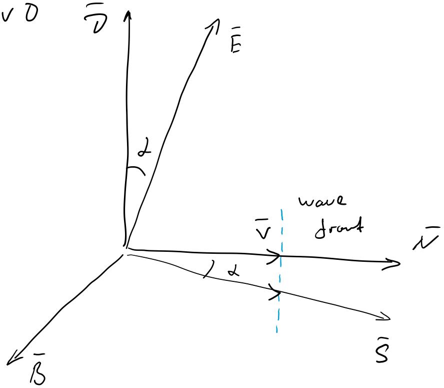
2) $[\bar\nabla \bar E] = -\frac{\partial \bar B}{\partial t}$

$$i[\bar k \bar E] = i\omega \bar B \qquad \boxed{[\bar N \bar E] = v \bar B}$$

What conclusions can we draw from this?

$$\varepsilon_0 c^2 [\bar N \bar B] = -v \bar D$$

$$[\bar N \bar E] = v \bar B$$



From first equation it is clear $\bar D \perp \bar B$ and $\bar D \perp \bar N$

From second equation $\bar B \perp \bar N$ and $\bar B \perp \bar E$.

That means three vectors $\bar D$, $\bar E$ and $\bar N$ are in one plane, with $\bar D \nparallel \bar E$.

It means, that in anisotropic media e/m waves are transverse for $\bar{B}$ and $\bar{D}$, but not for $\bar{E}$.

Since energy is carried along Poynting vector

$$S = [\bar{E}\bar{B}]$$

Let's introduce elementary vector

$$\bar{s} = \frac{[\bar{E}\bar{B}]}{EB} \qquad \bar{s} \perp \bar{E}$$ – ray vector

Angle between $\bar{D}$ and $\bar{E}$ is the same as $\bar{N}$ and $\bar{s}$:

$$\widehat{\bar{D}\bar{E}} = \widehat{\bar{N}\bar{s}} = \alpha$$

Hence we have:

1) Surface of equal phase propagate along $\bar{N}$ with speed v (phase velocity)

2) Speed of propagation of this plane along $\bar{s}$ is called ray velocity.

$$v = u\cos\alpha = u(\bar{N}\bar{s})$$

What is our goal now?

Our goal is to determine any velocity u in direction of $\bar{s}$.

Let's start from

$$\begin{cases} [\bar{N}\bar{E}] = v\bar{B} \\ \varepsilon_0 c^2[\bar{N}\bar{B}] = -v\bar{D} \end{cases}$$

multiplied by s on the left

$$\begin{cases} [\bar{s}[\bar{N}\bar{E}]] = v[\bar{s}\bar{B}] \\ \varepsilon_0 c^2[\bar{s}[\bar{N}\bar{B}]] = -v[\bar{s}\bar{D}] \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \bar{N}\underset{0}{(\cancel{\bar{s}\bar{E}})} - \bar{E}\overset{\cos\alpha}{(\bar{s}\bar{N})} = v[\bar{s}\bar{B}] \\ \varepsilon_0 c^2\left(\bar{N}\underset{0}{(\cancel{\bar{s}\bar{B}})} - \bar{B}(\bar{s}\bar{N})\right) = -v[\bar{s}\bar{D}] \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \bar{E} = -u[\bar{s}\bar{B}] \\ \varepsilon_0 c^2\bar{B} = u[\bar{s}\bar{D}] \end{cases} \qquad v = u\cos\alpha = u(\bar{N}\bar{s})$$

Let's exclude $\bar{B}$:

$$\bar{E} = \frac{1}{\varepsilon_0}\left(\frac{u}{c}\right)^2 \left[\bar{s}\left[\bar{s}\,\bar{D}\right]\right]$$

The remaining thing is to open double vector multiplication

$$\boxed{\varepsilon_0\left(\frac{c}{u}\right)^2 \bar{E} + \bar{s}(\bar{s}\,\bar{D}) - \bar{D} = 0}$$

This formula indicate that if we know $\bar{S}$, then we can determine ray velocity in the crystal. Important to notice, this velocity is different for different polarizations.